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A new two-parameter integrable model of strongly correlated fermions with quantum superalgebra symmetry

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Abstract. A new two-parameter integrable model with quantum superalgebra $U_q[gl(3|1)]$ symmetry is proposed, which is an eight-state fermions model with correlated single-particle and pair hoppings as well as uncorrelated triple-particle hopping. The model is solved and the Bethe ansatz equations are obtained.

Integrable strongly correlated fermion systems (see, e.g. [1]) have been an important subject of research activity (see e.g. [2, 3]). In a recent paper [4], we proposed an integrable correlated fermion model which is an eight-state version of the supersymmetric U model [5]. The latter has been extensively investigated by many authors [6]. By eight-state we mean that at a given lattice site j of the length L there are eight possible fermion states:

$$\begin{array}{llll} |0\rangle & c_{j,1}^{\dagger}|0\rangle & c_{j,2}^{\dagger}|0\rangle & c_{j,3}^{\dagger}|0\rangle \\ c_{j,1}^{\dagger}c_{j,2}^{\dagger}|0\rangle & c_{j,1}^{\dagger}c_{j,3}^{\dagger}|0\rangle & c_{j,2}^{\dagger}c_{j,3}^{\dagger}|0\rangle & c_{j,1}^{\dagger}c_{j,2}^{\dagger}c_{j,3}^{\dagger}|0\rangle \end{array} \tag{1}$$

where $c_{j,\alpha}^{\dagger}$ ($c_{j,\alpha}$) denotes a fermionic creation (annihilation) operator which creates (annihilates) a fermion of species $\alpha = 1, 2, 3$ at site j; these operators satisfy the anti-commutation relations given by $\{c_{i,\alpha}^{\dagger}, c_{j,\beta}\} = \delta_{ij}\delta_{\alpha\beta}$. The eight-state supersymmetric U model contains one free parameter and has Lie superalgebra $gl(3|1)$ as its symmetry.

In this paper, we promote the Lie superalgebra $gl(3|1)$ symmetry of the model to the quantum superalgebra $U_q[gl(3|1)]$ symmetry, thus giving a new integrable model of strongly correlated fermion with two free parameters. We then solve the two-parameter model by the coordinate space Bethe ansatz method and derive the Bethe ansatz equations.

The Hamiltonian for our new model on a periodic lattice reads

$$H(g,\kappa) = \sum_{j=1}^{L} H_{j,j+1}(g,\kappa)$$

$$H_{j,j+1}(g,\kappa) = -\sum_{\alpha}(c_{j,\alpha}^{\dagger}c_{j+1,\alpha} + \text{h.c.}) \exp\left\{ -\frac{1}{2}(\eta+\kappa)\sum_{\beta(\neq\alpha)} n_{j+\theta(\beta-\alpha),\beta} \right.$$
$$\left. -\frac{1}{2}(\eta-\kappa)\sum_{\beta(\neq\alpha)} n_{j+1-\theta(\beta-\alpha),\beta} + \frac{\zeta}{2}\sum_{\beta\neq\gamma(\neq\alpha)}(n_{j,\beta}n_{j,\gamma} + n_{j+1,\beta}n_{j+1,\gamma}) \right\}$$

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$$
\begin{aligned}
& -\frac{\sinh\kappa}{2\sinh\kappa(g+1)}\sum_{\alpha\neq\beta\neq\gamma}(c^\dagger_{j,\alpha}c^\dagger_{j,\beta}c_{j+1,\beta}c_{j+1,\alpha}+\text{h.c.}) \\
& \times\exp\left\{-\left(\frac{\xi}{2}-\operatorname{sign}(\gamma-2)\kappa\right)n_{j,\gamma}-\left(\frac{\xi}{2}+\operatorname{sign}(\gamma-2)\kappa\right)n_{j+1,\gamma}\right\} \\
& -\frac{2\cosh\kappa\sinh^2\kappa}{\sinh\kappa(g+1)\sinh\kappa(g+2)}(c^\dagger_{j,1}c^\dagger_{j,2}c^\dagger_{j,3}c_{j+1,3}c_{j+1,2}c_{j+1,1}+\text{h.c.}) \\
& +\mathrm{e}^{\kappa g}n_j+\mathrm{e}^{-\kappa g}n_{j+1}-\frac{\sinh\kappa}{2\sinh\kappa(g+1)}\sum_{\alpha\neq\beta}(n_{j,\alpha}n_{j,\beta}+n_{j+1,\alpha}n_{j+1,\beta}) \\
& +\frac{2\cosh\kappa(g+1)\sinh^2\kappa}{\sinh\kappa(g+1)\sinh\kappa(g+2)}(n_{j,1}n_{j,2}n_{j,3}+n_{j+1,1}n_{j+1,2}n_{j+1,3})
\end{aligned}
\tag{2}
$$

where g, κ are two free parameters, $n_j = n_{j,1} + n_{j,2} + n_{j,3}$ with $n_{j,\alpha} = c^\dagger_{i,\alpha}c_{j,\alpha}$ being the number operator for the fermion of species α at site j, $\theta(\beta-\alpha)$ is a step function of $(\beta-\alpha)$ and

$$
\eta = -\ln\frac{\sinh\kappa g}{\sinh\kappa(g+1)} \qquad \zeta = \frac{1}{2}\ln\frac{\sinh^2\kappa(g+1)}{\sinh\kappa g\sinh\kappa(g+2)} \qquad \xi = -\ln\frac{\sinh\kappa g}{\sinh\kappa(g+2)}. \tag{3}
$$

The model contains correlated single-particle and pair hoppings, uncorrelated triple-particle hopping, and two- and three-particle hoppings on site interactions. As is seen below, the chemical potential terms are essential for the model to have the quantum superalgebra symmetry. It should be noted that for the periodic lattice chemical potentials can be dropped from the Hamiltonian without affecting the integrability. However, they play an essential role for an open lattice. As we demonstrated in [7], without these terms, the boundary system would not be solvable by the coordinate Bethe ansatz method for a large class of integrable boundary conditions [7, 8].

Some remarks are in order. For the choice of $\kappa = 0$, the above model becomes the eight-state supersymmetric U model proposed in [4], whereas in the limit of $\eta = \kappa$ (first discarding the chemical potential terms in the Hamiltonian and then taking the limit), it reduces to the fermion model introduced in [9]:

$$
H(g) = -\sum_{j=1}^{L}\sum_{\alpha}(c^\dagger_{j,\alpha}c_{j+1,\alpha}+\text{h.c.})\exp\left\{-\eta\sum_{\beta(\neq\alpha)}n_{j+\theta(\beta-\alpha),\beta}\right\} \tag{4}
$$

whose integrability was established in [10].

The model defined by (2) is $U_q[gl(3|1)]$ supersymmetric and is exactly solvable on the one-dimensional periodic lattice. This is because the local Hamiltonian $H_{j,j+1}(g,\kappa)$ is actually derived through the quantum inverse scattering method using a $U_q[gl(3|1)]$ invariant R-matrix. To show this, we denote the generators of $U_q[gl(3|1)]$ by E^μ_ν, $\mu, \nu = 1, 2, 3, 4$ with grading $[1] = [2] = [3] = 0, [4] = 1$. In a typical eight-dimensional representation $V(\Lambda)$ of $U_q[gl(3|1)]$, the highest weight $\Lambda = (0, 0, 0|g)$ itself of the representation depends on the free parameter g, thus giving rise to a one-parameter family of inequivalent irreps. Let $\{|x\rangle\}_{x=1}^8$ denote an orthonormal basis with $|1\rangle, |5\rangle, |6\rangle, |7\rangle$ even (bosonic) and $|2\rangle, |3\rangle, |4\rangle, |8\rangle$ odd (fermionic). Then the simple generators $\{E^\mu_\mu\}_{\mu=1}^4$ and $\{E^\mu_{\mu+1}, E^{\mu+1}_\mu\}_{\mu=1}^3$ are 8×8 supermatrices of the form

$E^1_2 = |3\rangle\langle 4| + |5\rangle\langle 6|$
$E^2_1 = |4\rangle\langle 3| + |6\rangle\langle 5|$

$$
\begin{aligned}
E^1_1 &= -|4\rangle\langle 4| - |6\rangle\langle 6| - |7\rangle\langle 7| - |8\rangle\langle 8| \\
E^2_3 &= |2\rangle\langle 3| + |6\rangle\langle 7| \\
E^3_2 &= |3\rangle\langle 2| + |7\rangle\langle 6| \\
E^2_2 &= -|3\rangle\langle 3| - |5\rangle\langle 5| - |7\rangle\langle 7| - |8\rangle\langle 8| \\
E^3_4 &= \sqrt{[g]_q}|1\rangle\langle 2| + \sqrt{[g+1]_q}(|3\rangle\langle 5| + |4\rangle\langle 6|) + \sqrt{[g+2]_q}|7\rangle\langle 8| \\
E^4_3 &= \sqrt{[g]_q}|2\rangle\langle 1| + \sqrt{[g+1]_q}(|5\rangle\langle 3| + |6\rangle\langle 4|) + \sqrt{[g+2]_q}|8\rangle\langle 7| \\
E^3_3 &= -|2\rangle\langle 2| - |5\rangle\langle 5| - |6\rangle\langle 6| - |8\rangle\langle 8| \\
E^4_4 &= g|1\rangle\langle 1| + (g+1)(|2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|) \\
&\qquad +(g+2)(|5\rangle\langle 5| + |6\rangle\langle 6| + |7\rangle\langle 7|) + (g+3)|8\rangle\langle 8|
\end{aligned}
\tag{5}
$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$.

$U_q(gl(3|1))$ is a graded Hopf algebra with coproduct given by

$$
\begin{aligned}
\Delta(E^\mu_\mu) &= I \otimes E^\mu_\mu + E^\mu_\mu \otimes I \qquad \mu = 1,2,3,4 \\
\Delta(E^\mu_{\mu+1}) &= E^\mu_{\mu+1} \otimes q^{\frac{1}{2}(E^\mu_\mu - (-1)^{[\mu+1]}E^{\mu+1}_{\mu+1})} + q^{-\frac{1}{2}(E^\mu_\mu - (-1)^{[\mu+1]}E^{\mu+1}_{\mu+1})} \otimes E^\mu_{\mu+1} \\
\Delta(E^{\mu+1}_\mu) &= E^{\mu+1}_\mu \otimes q^{\frac{1}{2}(E^\mu_\mu - (-1)^{[\mu+1]}E^{\mu+1}_{\mu+1})} + q^{-\frac{1}{2}(E^\mu_\mu - (-1)^{[\mu+1]}E^{\mu+1}_{\mu+1})} \otimes E^{\mu+1}_\mu \qquad \mu = 1,2,3.
\end{aligned}
\tag{6}
$$

Associated with the eight-dimensional representation, there is a $U_q[gl(3|1)]$-invariant R-matrix which satisfies the graded Yang–Baxter equation. The R-matrix is given by

$$
\check{R}(u) = \check{P}_1 + \langle 2g\rangle\check{P}_2 + \langle 2g\rangle\langle 2g+2\rangle\check{P}_3 + \langle 2g\rangle\langle 2g+2\rangle\langle 2g+4\rangle\check{P}_4 \tag{7}
$$

where $\langle a\rangle = (1 - q^{u+a})/(q^u - q^a)$ and $\check{P}_k : \check{P}_k[V(\Lambda)\otimes V(\Lambda)] = V(\Lambda_k), k = 1, 2, 3, 4,$ are four projection operators which we are now going to construct. $\check{P}_1$ and $\check{P}_4$ are

$$
\check{P}_1 = \sum_{k=1}^{8} |\Psi^1_k\rangle\langle\Psi^1_k| \qquad \check{P}_4 = \sum_{k=1}^{8} |\Psi^4_k\rangle\langle\Psi^4_k|. \tag{8}
$$

Throughout this paper,

$$
\langle\Psi^a_k| = (|\Psi^a_k\rangle)^\dagger \qquad (|x\rangle\otimes|y\rangle)^\dagger = (-1)^{[|x\rangle][|y\rangle]}\langle y|\otimes\langle x| \tag{9}
$$

with $[|x\rangle] = 0$ for even (bosonic) $|x\rangle$ and $[|x\rangle] = 1$ for odd (fermionic) $|x\rangle$. $|\Psi^1_k\rangle, |\Psi^4_k\rangle, k = 1, 2, \ldots, 8$ are given by

$$
\begin{aligned}
|\Psi^1_1\rangle &= |1\rangle\otimes|1\rangle \\
|\Psi^1_i\rangle &= \frac{1}{\sqrt{q^g + q^{-g}}}(q^{\frac{g}{2}}|\mathrm{i}\rangle\otimes|1\rangle + q^{-\frac{g}{2}}|1\rangle\otimes|\mathrm{i}\rangle) \qquad i = 2,3,4 \\
|\Psi^1_i\rangle &= \frac{1}{\sqrt{(q^g + q^{-g})[2g+1]_q}}[\sqrt{[g+1]_q}(q^g|\mathrm{i}\rangle\otimes|1\rangle + q^{-g}|1\rangle\otimes|\mathrm{i}\rangle) \\
&\qquad +\sqrt{[g]_q}(q^{\frac{1}{2}}|2\rangle\otimes|\mathrm{i}-2\rangle - q^{-\frac{1}{2}}|\mathrm{i}-2\rangle\otimes|2\rangle)] \qquad \mathrm{i} = 5,6 \\
|\Psi^1_7\rangle &= \frac{1}{\sqrt{(q^g + q^{-g})[2g+1]_q}}[\sqrt{[g+1]_q}(q^g|7\rangle\otimes|1\rangle + q^{-g}|1\rangle\otimes|7\rangle) \\
&\qquad +\sqrt{[g]_q}(q^{\frac{1}{2}}|3\rangle\otimes|4\rangle - q^{-\frac{1}{2}}|4\rangle\otimes|3\rangle)] \\
|\Psi^1_8\rangle &= \frac{1}{\sqrt{(q^g + q^{-g})(q^{g+1} + q^{-g-1})[2g+1]_q}}[\sqrt{[g]_q}(q^{-\frac{g}{2}+1}|2\rangle\otimes|7\rangle + q^{\frac{g}{2}-1}|7\rangle\otimes|2\rangle \\
&\qquad +q^{\frac{g}{2}+1}|5\rangle\otimes 4\rangle + q^{-\frac{g}{2}-1}|4\rangle\otimes|5\rangle - q^{-\frac{g}{2}}|3\rangle\otimes|6\rangle - q^{\frac{g}{2}}|6\rangle\otimes|3\rangle) \\
&\qquad +\sqrt{[g+2]_q}(q^{\frac{3g}{2}}|8\rangle\otimes|1\rangle + q^{-\frac{3g}{2}}|1\rangle\otimes|8\rangle)]
\end{aligned}
\tag{10}
$$

$$|\Psi_1^4\rangle = \frac{1}{\sqrt{(q^{g+1}+q^{-g-1})(q^{g+2}+q^{-g-2})[2g+3]_q}}[\sqrt{[g+2]_q}(q^{-\frac{g}{2}-2}|7\rangle\otimes|2\rangle$$
$$-q^{\frac{g}{2}+2}|2\rangle\otimes|7\rangle + q^{-\frac{g}{2}}|5\rangle\otimes|4\rangle - q^{\frac{g}{2}}|4\rangle\otimes|5\rangle + q^{\frac{g}{2}+1}|3\rangle\otimes|6\rangle$$
$$-q^{-\frac{g}{2}-1}|6\rangle\otimes|3\rangle) + \sqrt{[g]_q}(q^{\frac{3g}{2}+3}|1\rangle\otimes|8\rangle - q^{-\frac{3g}{2}-3}|8\rangle\otimes|1\rangle)]$$
$$|\Psi_2^4\rangle = \frac{1}{\sqrt{(q^{g+2}+q^{-g-2})[2g+3]_q}}[\sqrt{[g+1]_q}(q^{-g-2}|8\rangle\otimes|2\rangle + q^{g+2}|2\rangle\otimes|8\rangle)$$
$$+\sqrt{[g+2]_q}(q^{\frac{1}{2}}|5\rangle\otimes|6\rangle - q^{-\frac{1}{2}}|6\rangle\otimes|5\rangle)]$$
$$|\Psi_i^4\rangle = \frac{1}{\sqrt{(q^{g+2}+q^{-g-2})[2g+3]_q}}[\sqrt{[g+1]_q}(q^{-g-2}|8\rangle\otimes|\mathrm{i}\rangle + q^{g+2}|\mathrm{i}\rangle\otimes|8\rangle)$$
$$+\sqrt{[g+2]_q}(q^{\frac{1}{2}}|\mathrm{i}+2\rangle\otimes|7\rangle - q^{-\frac{1}{2}}|7\rangle\otimes|\mathrm{i}+2\rangle)] \qquad i=3,4$$
$$|\Psi_i^4\rangle = \frac{1}{\sqrt{q^{g+2}+q^{-g-2}}}(-q^{-\frac{g}{2}-1}|8\rangle\otimes|\mathrm{i}\rangle + q^{\frac{g}{2}+1}|\mathrm{i}\rangle\otimes|8\rangle) \qquad i=5,6,7$$
$$|\Psi_8^4\rangle = |8\rangle\otimes|8\rangle.$$

The construction of the projector $\check{P}_2$ is slightly involved. We first construct the basis vectors for $V(\Lambda_2)$. With the help of the coproduct formulae and after some algebraic calculations, we find

$$|\Psi_1^2\rangle = \frac{1}{\sqrt{q^g+q^{-g}}}(-q^{-\frac{g}{2}}|2\rangle\otimes|1\rangle + q^{\frac{g}{2}}|1\rangle\otimes|2\rangle)$$
$$|\Psi_2^2\rangle = \frac{1}{\sqrt{q^g+q^{-g}}}(-q^{-\frac{g}{2}}|3\rangle\otimes|1\rangle + q^{\frac{g}{2}}|1\rangle\otimes|3\rangle)$$
$$|\Psi_3^2\rangle = |2\rangle\otimes|2\rangle$$
$$|\Psi_4^2\rangle = \frac{1}{\sqrt{q^g+q^{-g}}}(-q^{-\frac{g}{2}}|4\rangle\otimes|1\rangle + q^{\frac{g}{2}}|1\rangle\otimes|4\rangle)$$
$$|\Psi_5^2\rangle = \frac{1}{\sqrt{q+q^{-1}}}(q^{-\frac{1}{2}}|2\rangle\otimes|3\rangle + q^{\frac{1}{2}}|3\rangle\otimes|2\rangle)$$
$$|\Psi_6^2\rangle = |3\rangle\otimes|3\rangle$$
$$|\Psi_7^2\rangle = \frac{1}{\sqrt{q+q^{-1}}}(q^{-\frac{1}{2}}|2\rangle\otimes|4\rangle + q^{\frac{1}{2}}|4\rangle\otimes|2\rangle)$$
$$|\Psi_8^2\rangle = \frac{1}{\sqrt{q+q^{-1}}}(q^{-\frac{1}{2}}|3\rangle\otimes|4\rangle + q^{\frac{1}{2}}|4\rangle\otimes|3\rangle)$$
$$|\Psi_9^2\rangle = |4\rangle\otimes|4\rangle$$
$$|\Psi_{10}^2\rangle = \frac{1}{\sqrt{q^{g+1}+q^{-g-1}}}(-q^{-\frac{g}{2}-1}|2\rangle\otimes|5\rangle + q^{\frac{g}{2}+1}|5\rangle\otimes|2\rangle)$$
$$|\Psi_{11}^2\rangle = \frac{1}{\sqrt{q^{g+1}+q^{-g-1}}}(-q^{-\frac{g}{2}-1}|3\rangle\otimes|5\rangle + q^{\frac{g}{2}+1}|5\rangle\otimes|3\rangle)$$
$$|\Psi_{12}^2\rangle = \frac{1}{\sqrt{q^{g+1}+q^{-g-1}}}(-q^{-\frac{g}{2}-1}|2\rangle\otimes|6\rangle + q^{\frac{g}{2}+1}|6\rangle\otimes|2\rangle)$$
$$|\Psi_{13}^2\rangle = \frac{1}{\sqrt{q^{g+1}+q^{-g-1}}}(-q^{-\frac{g}{2}-1}|4\rangle\otimes|6\rangle + q^{\frac{g}{2}+1}|6\rangle\otimes|4\rangle)$$

$$
\begin{aligned}
|\Psi^2_{14}\rangle &= \frac{1}{\sqrt{q^{g+1}+q^{-g-1}}}(-q^{-\frac{g}{2}-1}|3\rangle\otimes|7\rangle + q^{\frac{g}{2}+1}|7\rangle\otimes|3\rangle)\\
|\Psi^2_{15}\rangle &= \frac{1}{\sqrt{q^{g+1}+q^{-g-1}}}(-q^{-\frac{g}{2}-1}|4\rangle\otimes|7\rangle + q^{\frac{g}{2}+1}|7\rangle\otimes|4\rangle)\\
|\Psi^2_{16}\rangle &= \frac{1}{\sqrt{(q^g+q^{-g})[2g+1]_q}}[\sqrt{[g]_q}(q^{g+\frac{1}{2}}|2\rangle\otimes|3\rangle + q^{-g-\frac{1}{2}}|3\rangle\otimes|2\rangle)\\
&\qquad +\sqrt{[g+1]_q}(|1\rangle\otimes|5\rangle - |5\rangle\otimes|1\rangle)]\\
|\Psi^2_{17}\rangle &= \frac{1}{\sqrt{(q^g+q^{-g})[2g+1]_q}}[\sqrt{[g]_q}(q^{g+\frac{1}{2}}|2\rangle\otimes|4\rangle + q^{-g-\frac{1}{2}}|4\rangle\otimes|2\rangle)\\
&\qquad +\sqrt{[g+1]_q}(|1\rangle\otimes|6\rangle - |6\rangle\otimes|1\rangle)]\\
|\Psi^2_{18}\rangle &= \frac{1}{\sqrt{(q^g+q^{-g})[2g+1]_q}}[\sqrt{[g]_q}(q^{g+\frac{1}{2}}|3\rangle\otimes|4\rangle + q^{-g-\frac{1}{2}}|4\rangle\otimes|3\rangle)\\
&\qquad +\sqrt{[g+1]_q}(|1\rangle\otimes|7\rangle - |7\rangle\otimes|1\rangle)]\\
|\Psi^2_{19}\rangle &= \frac{1}{\sqrt{(q+q^{-1})(q^{g+1}+q^{-g-1})}}(q^{\frac{g}{2}+1}|6\rangle\otimes|3\rangle - q^{-\frac{g}{2}-1}|3\rangle\otimes|6\rangle\\
&\qquad +q^{\frac{g}{2}}|5\rangle\otimes|4\rangle - q^{-\frac{g}{2}}|4\rangle\otimes|5\rangle)]\\
|\Psi^2_{20}\rangle &= \frac{1}{\sqrt{(q+q^{-1})(q^{g+1}+q^{-g-1})}}(q^{\frac{g}{2}+1}|7\rangle\otimes|2\rangle - q^{-\frac{g}{2}-1}|2\rangle\otimes|7\rangle\\
&\qquad +q^{\frac{g}{2}}|6\rangle\otimes|3\rangle - q^{-\frac{g}{2}}|3\rangle\otimes|6\rangle)]\\
|\Psi^2_{21}\rangle &= \frac{1}{\sqrt{(q^{g+1}+q^{-g-1})[2g+3]_q}}[\sqrt{[g+1]_q}(q^{-\frac{1}{2}}|6\rangle\otimes|5\rangle - q^{\frac{1}{2}}|5\rangle\otimes|6\rangle)\\
&\qquad +\sqrt{[g+2]_q}(q^{g+1}|8\rangle\otimes|2\rangle + q^{-g-1}|2\rangle\otimes|8\rangle)]\\
|\Psi^2_{22}\rangle &= \frac{1}{\sqrt{(q^{g+1}+q^{-g-1})[2g+3]_q}}[\sqrt{[g+1]_q}(q^{-\frac{1}{2}}|7\rangle\otimes|5\rangle - q^{\frac{1}{2}}|5\rangle\otimes|7\rangle)\\
&\qquad +\sqrt{[g+2]_q}(q^{g+1}|8\rangle\otimes|3\rangle + q^{-g-1}|3\rangle\otimes|8\rangle)]\\
|\Psi^2_{23}\rangle &= \frac{1}{\sqrt{(q^{g+1}+q^{-g-1})[2g+3]_q}}[\sqrt{[g+1]_q}(q^{-\frac{1}{2}}|7\rangle\otimes|6\rangle - q^{\frac{1}{2}}|6\rangle\otimes|7\rangle)\\
&\qquad +\sqrt{[g+2]_q}(q^{g+1}|8\rangle\otimes|4\rangle + q^{-g-1}|4\rangle\otimes|8\rangle)]\\
|\Psi^2_{24}\rangle &= \frac{1}{\sqrt{(q^g+q^{-g})(q^{g+1}+q^{-g-1})[2g+1]_q}}[\sqrt{[g]_q}(q^{\frac{g}{2}+1}|2\rangle\otimes|7\rangle - q^{-\frac{g}{2}-1}|7\rangle\otimes|2\rangle\\
&\qquad +q^{\frac{3g}{2}+1}|5\rangle\otimes|4\rangle - q^{-\frac{3g}{2}-1}|4\rangle\otimes|5\rangle - q^{\frac{g}{2}}|3\rangle\otimes|6\rangle - q^{-\frac{g}{2}}|6\rangle\otimes|3\rangle)\\
&\qquad +\sqrt{[g+2]_q}(-q^{\frac{g}{2}}|8\rangle\otimes|1\rangle + q^{-\frac{g}{2}}|1\rangle\otimes|8\rangle)].
\end{aligned}
\tag{11}
$$

However, this basis for $V(\Lambda_2)$ is not orthogonal, that is $\langle\Psi^2_l|\Psi^2_m\rangle \neq \delta_{lm}$ for some l, m. To make it orthogonal, we define a metric matrix g_{lm} by

$$g_{lm} = \langle\Psi^2_l|\Psi^2_m\rangle \qquad l, m = 1, 2, \ldots, 24 \tag{12}$$

where $\langle\Psi^2_k|$ are defined by (9). We then form a dual basis by means of the metric matrix, with basis vectors given by

$$\langle\Psi^{2,l}| = \sum_{m=1}^{24} g^{lm}\langle\Psi^2_m| \tag{13}$$

where $g^{lm} \equiv (g^{-1})_{lm}$ is the inverse of the metric matrix g_{lm}. Then by construction, $\langle \Psi^{2,l} | \Psi^2_m \rangle = \delta_{lm}$ for all $l,\ m$, which implies that $\langle \Psi^{2,l}|$ are orthogonal to $|\Psi^2_m\rangle$ for all $l,\ m$. Thus the projector $\check{P}_2$ is given by

$$\check{P}_2 = \sum_{k=1}^{24} |\Psi^2_k\rangle\langle\Psi^{2,k}|. \tag{14}$$

Finally the projector $\check{P}_3$ is obtained through the relation $\check{P}_1 + \check{P}_2 + \check{P}_3 + \check{P}_4 = 1$.

On the L-fold tensor product space $V \otimes V \otimes \cdots \otimes V$ we denote $\check{R}(u)_{j,j+1} = I^{\otimes(j-1)} \otimes \check{R}(u) \otimes I^{\otimes(L-j-1)}$, and define the local Hamiltonian by

$$H^{\mathrm{R}}_{j,j+1}(g,q) = \frac{\mathrm{d}}{\mathrm{d}u}\check{R}_{j,j+1}(u)\bigg|_{u=0}. \tag{15}$$

We make the identifications:

$$\begin{aligned} &|1\rangle = |0\rangle \qquad |2\rangle = c^\dagger_{j,1}|0\rangle \qquad |3\rangle = c^\dagger_{j,2}|0\rangle \qquad |4\rangle = c^\dagger_{j,3}|0\rangle \\ &|5\rangle = c^\dagger_{j,1}c^\dagger_{j,2}|0\rangle \qquad |6\rangle = c^\dagger_{j,1}c^\dagger_{j,3}|0\rangle \qquad |7\rangle = c^\dagger_{j,2}c^\dagger_{j,3}|0\rangle \qquad |8\rangle = c^\dagger_{j,1}c^\dagger_{j,2}c^\dagger_{j,3}|0\rangle. \end{aligned} \tag{16}$$

Then by (8), (14), (11), (12), (9), (13) and (16), and after tedious but straightforward manipulation, one obtains, up to a constant,

$$H_{j,j+1}(g,\kappa) = \frac{2\sinh\kappa g}{\kappa} H^{\mathrm{R}}_{j,j+1} \qquad (g, q = \mathrm{e}^{-\kappa}). \tag{17}$$

This identity also shows that $H(g,\kappa)$ commutes with the generators (6) of $U_q[gl(3|1)]$, since the R-matrix $\check{R}(u)$ is a $U_q[gl(3|1)]$ invariant.

We now solve the system by means of the coordinate space Bethe ansatz technique. We assume the following wavefunction

$$\psi_{\alpha_1,\ldots,\alpha_N}(x_1,\ldots,x_N) = \sum_P \epsilon_P A_{\alpha_{Q_1},\ldots,\alpha_{Q_N}}(k_{P_{Q_1}},\ldots,k_{P_{Q_N}}) \exp\left(\mathrm{i}\sum_{j=1}^N k_{P_j}x_j\right) \tag{18}$$

where Q is the permutation of the N particles such that $1\langle x_{Q_1}\langle\cdots\langle x_{Q_N} \leqslant L$. Denote $X_Q = \{x_{Q_1}\langle\cdots\langle x_{Q_N}\}$. The coefficients $A_{\alpha_{Q_1},\ldots,\alpha_{Q_N}}(k_{P_{Q_1}},\ldots,k_{P_{Q_N}})$ from regions other than X_Q are connected with each other by elements of the two-particle S-matrix:

$$\begin{aligned} &S_{ij}(k_i,k_j)^{aa}_{aa} = 1 \qquad a = 1,2,3 \\ &S_{ij}(k_i,k_j)^{ab}_{ab} = \frac{\sin(\lambda_i - \lambda_j)}{\sin(\lambda_i - \lambda_j - \mathrm{i}\kappa)} \qquad a \neq b, a, b = 1,2,3 \\ &S_{ij}(k_i,k_j)^{ab}_{ba} = \mathrm{e}^{\mathrm{i}\,\mathrm{sign}(a-b)(\lambda_i-\lambda_j)}\frac{\sin\mathrm{i}\kappa}{\sin(\lambda_i - \lambda_j - \mathrm{i}\kappa)} \qquad a, b = 1,2,3 \end{aligned} \tag{19}$$

where λ_j are suitable particle rapidities related to the quasimomenta k_j of the electrons by

$$k(\lambda) = 2\arctan(\coth c \tan\lambda) \tag{20}$$

where the parameter c is defined by

$$c = \frac{1}{4}\left\{\ln\left[\frac{\sinh\frac{1}{2}(\eta+\kappa)}{\sinh\frac{1}{2}(\eta-\kappa)}\right] - \kappa\right\}. \tag{21}$$

The periodicity condition for the system on the finite interval $(0, L)$ results in the Bethe equations for the set of N momenta $k_j : \exp(\mathrm{i}k_jL) = T_j,\ j = 1,\ldots,N$, where

$$T_j = S_{j,j+1}(k_j,k_{j+1})\cdots S_{j,N}(k_j,k_N)S_{j,1}(k_j,k_1)\cdots S_{j,j-1}(k_j,k_{j-1}),\ j = 1,\ldots,N. \tag{22}$$

The meaning of T_j is the scattering matrix of the jth particle on the other $(N-1)$ particles. So now the problem is to diagonalize T_j to arrive at a system of scalar equations. It can be shown that $T_j = \tau(\lambda = k_j)$, where

$$\tau(\lambda) = \mathrm{tr}_0[S_{0,1}(\lambda - k_1) \cdots S_{0,N}(\lambda - k_N)] \tag{23}$$

is the transfer matrix of the inhomogeneous $U_q[gl(3)]$-spin magnet of N sites. The commutativity of the transfer matrix for different values of the spectral parameter λ implies that T_j, $j = 1, \ldots, N$ can be diagonalized simultaneously. The Bethe ansatz equations are written in terms of the rapidities $\Lambda_\sigma^{(1)}$, $\Lambda_\sigma^{(2)}$ and λ_σ

$$\begin{aligned}
&\mathrm{e}^{\mathrm{i}k_j L} = \prod_{\sigma=1}^{M_1} \frac{\sin(\lambda_j - \Lambda_\sigma^{(1)} - \mathrm{i}\kappa/2)}{\sin(\lambda_j - \Lambda_\sigma^{(1)} + \mathrm{i}\kappa/2)} \\
&\prod_{j=1}^{N} \frac{\sin(\Lambda_\sigma^{(1)} - \lambda_j + \mathrm{i}\kappa/2)}{\sin(\Lambda_\sigma^{(1)} - \lambda_j - \mathrm{i}\kappa/2)} = -\prod_{\rho=1}^{M_1} \frac{\sin(\Lambda_\sigma^{(1)} - \Lambda_\rho^{(1)} + \mathrm{i}\kappa)}{\sin(\Lambda_\sigma^{(1)} - \Lambda_\rho^{(1)} - \mathrm{i}\kappa)} \prod_{\rho=1}^{M_2} \frac{\sin(\Lambda_\sigma^{(1)} - \Lambda_\rho^{(2)} - \mathrm{i}\kappa/2)}{\sin(\Lambda_\sigma^{(1)} - \Lambda_\rho^{(2)} + \mathrm{i}\kappa/2)} \\
&\qquad \sigma = 1, \ldots, M_1 \\
&\prod_{\rho=1}^{M_1} \frac{\sin(\Lambda_\gamma^{(2)} - \Lambda_\rho^{(1)} + \mathrm{i}\kappa/2)}{\sin(\Lambda_\gamma^{(2)} - \Lambda_\rho^{(1)} - \mathrm{i}\kappa/2)} = -\prod_{\rho=1}^{M_2} \frac{\sin(\Lambda_\gamma^{(2)} - \Lambda_\rho^{(2)} + \mathrm{i}\kappa)}{\sin(\Lambda_\gamma^{(2)} - \Lambda_\rho^{(2)} - \mathrm{i}\kappa)} \qquad \gamma = 1, \ldots, M_2.
\end{aligned} \tag{24}$$

The energy of the system in the state corresponding to the sets of solutions $\{\Lambda_\sigma^{(1)}, \Lambda_\sigma^{(2)}\}$ and $\{\lambda_\sigma\}$ is (up to an additive constant, which we drop) $E = -2\sum_{j=1}^{N} \cos k_j$.

Therefore, we have presented a new two-parameter integrable model which is an eight-state supersymmetric fermion model with correlated single-particle and pair hoppings as well as uncorrelated triple-particle hopping. We have solved the model by the coordinate Bethe ansatz method and derived the Bethe ansatz equations. There are many areas still requiring attention in this new model. One of them is to incorporate integrable boundary conditions into the model. We hope to report results on this aspect in future publications.

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